SPINOR AS AN INVARIANT

(SPINOR KAK INVARIANTNYI OB'EKT)

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1. Representation of spinors in terms of tensor systems. In quantum mechanics, some of the elementary particles can be described in terms of several wave functions, which represent a spinor in a three- or four-dimensional space. Spinors can also be used as generalized parameters in constructing the models of continuous media.

When we regard a spinor as a linear geometrical entity, then we can define it only on the orthogonal group of transformations of coordinates. Nevertheless, spinor equations can be written in an equivalent form which lends itself to investigation in arbitrary curvilinear coordinate systems.

1. Basic definitions. We shall first consider spinors in the four-dimensional Minkowski space R_4, referred to the original coordinate system x^k. Coordinate x^4 shall be assumed to be complex. Let γ_1, γ_2, γ_3 and γ_4 be Dirac matrices which, by definition, satisfy

$$\gamma_i\gamma_j + \gamma_j\gamma_i = 2\delta_{ij}J \tag{1.1}$$

where δ_{ij} is Kronecker delta and J is a unit matrix.

If the system γ_i (i = 1, 2, 3, 4) is a solution of (1.1), then we can easily see that the system $T\gamma_i T^{-1}$ will also be a solution of (1.1) for any nondegenerate matrix T. Any two solutions γ_i and γ_i° are connected by the relation $\gamma_i^\circ = T\gamma_i T^{-1}$ where the matrix T is suitably defined [1].

We shall use the following set of Hermitian matrices γ_i

$$\gamma_1 = \left\|\begin{matrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{matrix}\right\|, \quad \gamma_2 = \left\|\begin{matrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{matrix}\right\|, \quad \gamma_3 = \left\|\begin{matrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{matrix}\right\|, \quad \gamma_4 = \left\|\begin{matrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{matrix}\right\| \tag{1.2}$$

Let $L = \|l_q{}^p\|$ be a Lorentz transformation of space R_4, and let a unimodular fourth-order matrix S defined by

$$\gamma_i = l_i{}^p S\gamma_p S^{-1}$$

correspond to each Lorentz transformation L.

Set of matrices S corresponding to the group of Lorentz transformations L, will also be a group, and will generate a linear representation of the group L, which shall be called a spinor representation. Finally, we shall call $\Psi = \{\psi^i\}$ whose components ψ^i are defined with accuracy of up to the change of sign and which transforms according to S, a spinor of first rank in the space R_4.

It can be shown that the group of spinor transformations S cannot be considered as a subgroup of some group effecting the representation of a complete affine group of coordinate transformations (see Appendix). Obviously, if the system of matrices γ_i is a solution of (1.1), then transposed matrices γ_i^* also satisfy (1.1), consequently there exist a matrix C such, that

$$\gamma_i^* = C\gamma_i C^{-1}, \qquad \det C = 1$$

Covariant components of the spinor ψ_k are given by

$$\psi_k = e_{ki}\psi^i \qquad (E = \| e_{ki} \| = C\gamma_5, \;\; \gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4)$$

2. Representation of spinors in terms of complex tensors. We know [2] that the spinor Ψ in the space R_4 is equivalent to the tensor field Λ composed of a complex vector C_i and a complex antisymmetric tensor of second rank C_{pq} which satisfy six independent algebraic equations. C_i and C_{pq} are given by

$$C_i = (E\gamma_i)_{mn}\,\psi^m\psi^n, \qquad C_{pq} = {}^1/_2 E\,(\gamma_p\gamma_q - \gamma_q\gamma_p)_{mn}\,\psi^m\psi^n \tag{1.3}$$

For the matrices (1.2) we have

$$E = \gamma_4\gamma_2 = \begin{Vmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{Vmatrix}$$

Components of ψ^k and the tensor field Λ are connected by Formula

$$\psi^k = \frac{\psi^{nk}}{\pm\sqrt{\psi^{nn}}} \tag{1.4}$$

Here $\psi^{nk} = \psi^n\psi^k$ denote the components of a spinor which is algebraically equivalent to Λ. Components ψ^{nk} can be obtained in terms of the components of C_i and C_{pq} by means of Formulas (1.3).

Components of the tensors C_i and C_{pq} satisfy the following invariant equations [3]:

$$C^iC_i = 0, \quad C^{ik}C_{ik} = 0, \quad C^{[ik}C^{pq]} = 0, \quad C^iC_{ik} = 0, \quad C^iC^k + C^{ip}C_p{}^k = 0, \quad C^{[i}C^{pq]} = 0 \tag{1.5}$$

Here square brackets around the superscripts denote the alternation over these indices. Fifth equation of (1.5) defines, with accuracy of up to the change of sign, components of the vector C_k in terms of the components of C_{pq}

$$C_k = \frac{iC_{kn}C_p{}^n}{\pm\sqrt{C_{pn}C_p{}^n}}$$

Out of all equations of (1.5),(1.5.2),(1.5.3) and four equations out of (1.5.5) are independent.

By (1.4), every spinor equation has an equivalent expression in terms of the components of C_i and C_{pq}.

3. Representation of spinors in terms of real tensors. Components of the spinor ψ^k may form not only a complex field $\Lambda = \{C_i, C_{pq}\}$, but also a real tensor field $\Omega = \{\Omega, j^k, M^{ik}, S^{ijk}, N^{ijkl}\}$, satisfying nine independent algebraic equations. Components of the tensors $\Omega, j^k, M^{ik}, S^{ijk}$ and N^{ijkl} can be defined [4] thus (*)

*) It can be shown that an orthogonal coordinate system exists, in which the components of tensors C^i, C^{pq}, j^p, M^{pq} and $S_k = {}^1/_6\varepsilon_{kijl}S^{ijl}$ have the form (see Appendix)

$$j^k = (0, 0, 0, i\rho) \qquad S^k = (0, 0, i\rho, 0), \qquad C^k = (-i\rho, 0, 0, 0)$$

$$M^{ij\cdot} = \begin{Vmatrix} 0 & -\Omega & 0 & 0 \\ \Omega & 0 & 0 & 0 \\ 0 & 0 & 0 & N \\ 0 & 0 & -N & 0 \end{Vmatrix}, \quad C^{ij} = \begin{Vmatrix} 0 & 0 & N-i\,\Omega \\ 0 & 0 & iN & \Omega \\ -N-i\,N & 0 & 0 \\ i\Omega - & \Omega & 0 & 0 \end{Vmatrix}$$

$$\Omega = \gamma^4_{mn}\bar{\psi}^m\psi^n, \qquad J_p = i(\gamma_4\gamma_p)_{mn}\bar{\psi}^m\psi^n, \qquad M^{pq} = i(\gamma_4\gamma_p\gamma_q)_{mn}\bar{\psi}^m\psi^n$$
$$S_{ijk} = (\gamma_4\gamma_i\gamma_j\gamma_k)_{mn}\bar{\psi}^m\psi^n, \qquad N_{ijkl} = (\gamma_4\gamma_i\gamma_j\gamma_k\gamma_l)_{mn}\bar{\psi}^m\psi^n, \qquad i \neq j \neq k \neq l \tag{1.6}$$

Components of the tensors Ω, j^p, M^{ij}, S^{ijk} and N^{ijkl} satisfy a number of invariant equations, the following of which are known [5, 6 and 7]:

$$J^kJ_k = -\Omega^2 + {}^1/_{24}N_{ijkl}N^{ijkl}, \qquad {}^1/_6S_{ijk}S^{ijk} = -\Omega^2 + {}^1/_4N_{ijkl}N^{ijkl}$$
$$\delta^{ijkl}_{pqrn}S_{ijk}j_l = 0, \qquad {}^1/_2M^{ij}M_{ij} = \Omega^2 + {}^1/_{24}N_{ijkl}N^{ijkl}$$
$$\delta^{pqrn}_{ijkl}M^{ij}M^{kl} = 8\Omega N^{pqrn}, \qquad M_{kp}j^k = {}^1/_6N_{ijkp}S^{ijk}$$
$$S^{klm}j_k = \Omega M^{lm} + {}^1/_2N^{lmkr}M_{kr} \tag{1.7}$$

It is also easy to obtain the following equations:

$$\delta^{pqr}_{i\,jk}M^i{}_nS^{njk} = 2N^{pqrn}j_n, \qquad M_{pq}S^{pqk} = -2\Omega j^k$$
$${}^1/_2S^{ipq}S_{pqj} = j^ij_j + M^{ik}M_{kj} + {}^1/_{24}N_{pqkl}N^{pqkl}g_j{}^i$$
$$j_nS^{ijk} - {}^1/_6\delta^{l\,ijk}_{npqr}j_lS^{pqr} + M_{np}N^{pijk} = {}^1/_2\delta^{i\,jk}_{npq}\,\Omega M^{pq} \tag{1.8}$$

Tensor δ^{pqr}_{ijk} is defined as follows:

$$\delta^{pqr}_{i\,jk} = \delta_i{}^p\delta_j{}^q\delta_k{}^r - \delta_i{}^p\delta_j{}^r\delta_k{}^q + \delta_i{}^q\delta_j{}^r\delta_k{}^p - \delta_i{}^q\delta_j{}^r\delta_k{}^p + \delta_i{}^r\delta_j{}^p\delta_k{}^q - \delta_i{}^r\delta_j{}^q\delta_k{}^p$$

and the tensor δ^{pqrn}_{ijkl} can be found in the analogous manner. We can also assume [8] that Equations (1.7.1),(1.73),(1.7.5) and (1.7.6) are independent.

We know [2] that the tensor field Ω defines the components of the spinor ψ^k with accuracy of up to the factor whose modulus is unity. The connection between the components of the spinor and Ω is given by

$$\psi^k = \frac{\psi^{n\cdot k}}{\pm\sqrt{\psi^{n\cdot n}}}\ \exp\,(i\varphi) \tag{1.9}$$

Here $\psi^{n\cdot k} = \bar{\psi}^n\psi^k$ is algebraically equivalent to Ω .

Components of $\psi^{n\cdot k}$ can be found in terms of Ω, j^p, M^{ij}, S^{ijk} and N^{ijkl} from (1.6) and φ is an arbitrary real number.

Since the relationship (1.9) between the components of the spinor and Ω exists, arbitrary spinor equations can be written in an equivalent form in terms of components of Ω and of the phase φ . Eliminating the phase φ from these equations we can obtain tensor equations in terms of the components of Ω . To complete the resulting set of tensor equations, nine independent algebraic equations (1.7) must of course be added.

Closed system of equations which we have obtained, will not be equivalent to the initial spinor equations. Nevertheless, this shortcoming is not important when it comes to consider physical aspects of the phenomena described by spinor equations, since only the tensors (1.6) have a direct physical meaning and all physical magnitudes (taking into account φ and the tensors Ω, j^k, M^{ij}, S^{ijk} and N^{ijkl} from the initial equations) can be expressed in terms of these tensors.

Tensors (1.3) and (1.6) are also algebraically interrelated and, in particular, the following relations exist:

$$4\Omega^2 = \bar{C}^iC_i - {}^1/_2C_{pq}\bar{C}^{pq}, \qquad 4\Omega j^q = i(C_p\bar{C}^{qp} - \bar{C}_pC^{qp})$$
$$4\Omega M^{pq} = i\delta^{pq}_{i\,j}(\bar{C}^iC^j - C_r{}^i\bar{C}^{rj}), \qquad 8\Omega S^{ijk} = \delta^{i\,jk}_{pqr}(\bar{C}^pC^{qr} + C^p\bar{C}^{qr})$$
$$16\Omega N^{ijkl} = \delta^{i\,jkl}_{pqrn}\bar{C}^{pq}C^{rn}, \qquad N_{ijkl}N^{ijkl} = -6(\bar{C}^iC_i + {}^1/_2\bar{C}_{ij}C^{ij}) \tag{1.10}$$
$$4j^pj^q = \bar{C}^pC^q + C^p\bar{C}^q - \bar{C}^{pk}C_k{}^q - C^{pk}\bar{C}_k{}^q - (\bar{C}^iC_i + {}^1/_2\bar{C}^{ij}C_{ij})g^{pq}$$

$$S^pS^q + j^p j^q + g^{pq}(\Omega^2 - 1/_{24}N^{ijkl}N_{ijkl}) = 1/_2(\bar{C}^pC^q + C^p\bar{C}^q)$$
$$j^qC_{pq} = i\Omega C_p, \qquad S_{pqr}\bar{C}^r = \Omega C_{pq} + 1/_2N_{pqrn}C^{rn}$$
$$M^{kp}C_k = i\Omega C^p, \qquad S^{ijk}C_{ij} = -2\Omega C^k$$
$$C_q j_p = -i\Omega C_{pq} - M_{pk}C^k{}_q, \qquad C_q j_p = i/2N_{knqp}C^{kn} - M_{qk}C^k{}_p$$

where $\bar{C}^{pq}$ denotes complex conjugates of C^{pq} with the change of sign of the components C^{4p} .

4. Two-component spinors in the Minkowski space. Spinors in the three-dimensional space. We know [4] that the transformation matrix S of the spinors corresponding to characteristic Lorentz transformation has, for the matrices γ_i given by Formula (1.2), the form

$$S = \begin{Vmatrix} \Sigma & 0 \\ 0 & \Sigma^{*-1} \end{Vmatrix}$$

Therefore, two component pairs $\{\psi^1\psi^2\}$ and $\{\psi^3\psi^4\}$ transform under the characteristic Lorentz group independently of each other. This fact allows us to consider, on the characteristic Lorentz group, not only four-component spinors, but also two-component spinors.

Everything that was derived above for the four-component spinors is true for the two-component spinors and corresponding formulas are obtained by putting $\psi^3 = \psi^4 = 0$.

These additional conditions lead to considerable simplification of the relationship between spinors and tensors, as in this case components of the vector C^i are identically equal to zero, while the tensor C^{pq} assumes a special form (in the orthogonal coordinate system), namely

$$C^{pq} = \begin{Vmatrix} 0 & -p_z & p_y & p_x \\ p_z & 0 & -p_x & p_y \\ -p_y & p_x & 0 & p_z \\ -p_x & -p_y & -p_z & 0 \end{Vmatrix}, \qquad p_x{}^2 + p_y{}^2 + p_z{}^2 = 0 \tag{1.11}$$

and satisfies an additional relation

$$C_{pq} = -1/_2\varepsilon_{pqij}C^{ij} \tag{1.12}$$

where ε_{pqij} is a unit, completely antisymmetric pseudo-tensor.

By (1.12) all the identities of (1.5) except (1.5.3) are satisfied. Therefore, two-component spinor in the Minkowski space is equivalent to the antisymmetric tensor C^{pq} satisfying the identities (1.5.3) and (1.12). Ω in this case consisits of an isotropic vector j^k. Equations (1.10) become

$$j^pj_p = 0, \qquad j^pC_{pq} = 0$$

Under three-dimensional spatial transformations, components p_x, p_y and p_z become the components of the pseudo-vector. Spinors in three-dimensional space can be considered as a particular case of two-component spinors in the four-dimensional space, therefore it is clear that a spinor in three-dimensional space is equivalent to an isotropic complex pseudo-vector. Existence of two-component spinors in four-dimensional space is closely related to the existence of an invariant submanifold $\{0, C^{pq}\}$. This means that, when fundamental spin tensors γ_i are chosen arbitrarily (and not only in the form of (1.2)), then a spinor space contains a two-dimensional subspace, which is invariant relative to the characteristic Lorentz group.

In order to write spinor equations in tensor form, we can also use, apart from (1.4) and (1.9), the following relation between the spinor and the quantities ψ^{pq} and $\psi^{p^{\cdot}q}$

$$\psi^k = \frac{\psi^{n^{\cdot}k}}{\pm\sqrt{\bar{\psi}^{nn}}} \tag{1.13}$$

With algebraic equations (1.10) taken into account, (1.13) allows us to obtain a closed system of tensor equations in terms of C_p, C_{pq}, Ω, j^p, M^{ij}, S^{ijk} and N. Equivalence of such a system of tensor equations to the initial system of spinor equations, follows from the previous argument.

Summing everything up, we can draw the following conclusions.

1. Four-component spinor in the Minkowski space is equivalent to the tensor field $\Lambda = \{C_1, C^{pq}\}$ which satisfies Equations (1.5).

Two-component spinor in the Minkowski space is equivalent to the antisymmetric tensor C_{pq} satisfying Equations

$$C^{[pq}C^{ij]} = 0, \qquad C_{pq} = -{}^1/_2\varepsilon_{pqij}C^{ij}$$

A spinor in a three-dimensional space is equivalent to an isotropic complex pseudo-vector.

2. An equivalent form of any spinor equation can be written in terms of components of tensors C_1 and C_{pq}.

3. Spinor equations can generate a closed system of equations in terms of components of tensors $\Omega, j^p, M^{ij}, S^{ijk}$ and N^{ijkl}.

2. Tensor forms of Dirac equations. 1. Dirac equations in terms of components off C_1 and C_{pq}. In the relativistic theory of electrons proposed by Dirac, equations which are established for four wave functions of the electron ψ^k, form a spinor of first rank in the Minkowski space. They can be written as

$$\gamma^k_{nm}\left(\frac{\partial\psi^m}{\partial x^k} + \frac{ie}{\hbar c}A_k\psi^m\right) + \frac{mc}{\hbar}\psi^n = 0 \tag{2.1}$$

where A_k is some vector potential of external electromagnetic fields, $x^4 = ict$, m and e are the mass and charge of the electron, γ^k are Dirac matrices, $\hbar$ is the Plank's constant and c is the velocity of light in vacuo.

Summation from 1 to 4 is performed over the indices k and m. Inserting the identity $\psi^m = \psi^{nm}/\pm\sqrt{\psi^{nn}}$, into (2.1), we easily obtain

$$\gamma^k_{nm}\left(\frac{\partial\psi^{\nu m}}{\partial x^k} - \frac{1}{2}\frac{\psi^{\nu m}}{\psi^{\nu\nu}}\frac{\partial\psi^{\nu\nu}}{\partial x^k}\right) + \left(\frac{ie}{\hbar c}A_k\gamma^k_{nm}\psi^{\nu m} + \frac{mc}{\hbar}\psi^{\nu n}\right) = 0 \tag{2.2}$$

in which the summation over ν is omitted. On contraction with $E_{mn}\psi^{m\nu}$ over n, the above equation gives

$$\left(\frac{\partial}{\partial x^k} + \frac{2ie}{\hbar c}A_k\right)C^k = 0 \tag{2.3}$$

Contraction of (2.2) with the matrices $(E\gamma_i)_{mn}\psi^{m\nu}$ over n, or alternatively, over ν with diagonal matrices J, γ_5, $\gamma_3\gamma_4$ and $\gamma_1\gamma_2$, yields

$$2C^{pq}\nabla_n C^{nk} + \delta^{pq}_{rn}(C^{ri}\nabla^k C_i{}^n + C^r\nabla^k C^n) + 4C^{pq}\left(\frac{ie}{\hbar c}A_n C^{nk} - \frac{mc}{\hbar}C^k\right) = 0$$

$$(p,\ q = 2{,}4,\ p,\ q = 1{,}4) \tag{2.4}$$

$$2\,C^p\nabla_n C^{nk} + (C^{pi}\nabla^k C_i - C_i\nabla^k C^{pi}) + 4C^p\left(\frac{ie}{\hbar c}A_n C^{nk} - \frac{mc}{\hbar}C^k\right) = 0 \quad (p = 1{,}2) \tag{2.5}$$

Obviously, these equations are valid for all sets of indices p and q, but they cease to be independent.

There are three independent equations with respect to k in (2.4) and (2.5).

Using the identities (1.5.1) and (1.5.2), we can reduce these equations to

$$C^{pq}\left(\nabla_n C^{nk} + \frac{2ie}{\hbar c}A_n C^{nk} - \frac{2mc}{\hbar}C^k\right) + C^{pn}\nabla^k C_n{}^q + C^p\nabla^k C^q = 0 \tag{2.6}$$

$$C^p\left(\nabla_n C^{nk} + \frac{2ie}{\hbar c}A_n C^{nk} - \frac{2mc}{\hbar}C^k\right) + C^{pn}\nabla^k C_n = 0 \tag{2.7}$$

Equation (2.3) and any two equations from (2.7) with one equation from (2.6), or two equations from each (2.6) and (2.7) (in k), form a system of equations equivalent to the Dirac system, and Dirac equations in spinor form can be obtained from them by reversing the procedure used previously to obtain tensor equations. Equivalence of these equations can only be violated during the process of multiplying (2.1) by $\sqrt{\psi^{\nu\nu}}$ and during the contraction of (2.2).

It can however be verified, that in both cases this only leads to the loss of the null solution and both, Dirac equations and the set (2.3),(2.6) and (2.7) have a null solution.

Contracting Equations (2.2) with various matrices γ^i, $\gamma^i\gamma^j, \ldots$ over the indices n and ν, we can obtain a number of other equations which follow from Dirac equations, but they will all be dependent on (2.6) and (2.7) and will not contribute anything towards the construction of a complete system of tensor equations.

Using the identity (1.13), we can write Dirac equations thus

$$\gamma_{nm}^{k}\left(\frac{\partial\psi^{\nu'm}}{\partial x^k}-\frac{1}{2}\frac{\psi^{\nu'm}}{\bar{\psi}^{\nu\nu}}\frac{\partial\bar{\psi}^{\nu\nu}}{\partial x^k}\right)+\left(\frac{ie}{\hbar c}A_k\gamma_{nm}^{k}\psi^{\nu'm}+\frac{mc}{\hbar}\psi^{\nu'n}\right)=0 \tag{2.8}$$

This, on contraction over n and ν with various matrices γ^i, yields equations in terms of components of tensors $C^p, C^{pq}, \Omega, j^p, \ldots, N^{ijkl}$. These equations are also equivalent to the Dirac system and, in particular, the following equations hold

$$\Omega\left(\frac{2mc}{\hbar}j^p-\nabla_k M^{kp}\right)+i\,[{}^1\!/_4\,(\bar{C}_i\nabla^p C^i-{}^1\!/_2\bar{C}_{ij}\nabla^p C^{ij})-\Omega\nabla^p\Omega-\frac{2e}{\hbar c}\;\Omega^2 A^p=0$$

$$C^n\left(\frac{2mc}{\hbar}j^p-\nabla_k M^{kp}-\frac{2e}{\hbar c}\Omega A^p\right)+i\nabla^p(\Omega C^n)-i/2\,(C^n\nabla^p\Omega-$$
$$-iC_k\nabla^p M^{kn}-iC^{nk}\nabla^p j_k-{}^1\!/_2 C_{ij}\nabla^p S^{ijn})=0 \tag{2.9}$$
$$\Omega\left(\nabla_k C^{kq}-\frac{2mc}{\hbar}C^q+\frac{2ie}{\hbar c}A_k C^{kq}\right)+i/2\,[-j_k\nabla^q C^k-{}^1\!/_2 M_{kp}\nabla^q C^{kp}]=0$$

2. Equations for the neutrino in tensor form . Relativistic equation for neutrino in the form given by Pauli, Lee and Yang, is

$$\sigma^k\frac{\partial}{\partial x^k}\psi=0$$

where σ^k are two-row matrices satisfying the relation

$$\bar{\sigma}^p\sigma^q+\sigma^q\;\bar{\sigma}^p=2g^{pq}J$$

Pauli-Lee-Yang equations can be obtained from Dirac equations (2.1) by putting $m=0, \psi^3=\psi^4=0, A_k=0$. Taking into account (3) of Section 1, we find from (2.6), that the tensor form of Pauli-Lee-Yang equations have the form

$$C^{pq}\nabla_n C^{nk}+C^{pn}\nabla^k C_n{}^q=0 \tag{2.10}$$

where C^{pq} is an antisymmetric tensor satisfying the identities (1.5.3) and (1.12) and which contains two independent equations with respect to k (2.10).

From (2.9) we also obtain equations in terms of components of j^k and C^{pq} ; they are:

$$j^r\nabla_k C^{pk}-j^k\nabla^p C_k{}^r=0 \tag{2.11}$$

3. Dirac equations in terms of components of tensors $\Omega, j^p, M^{pq}, S^{pqr}$ and N^{pqrn}. System of quasilinear equations (2.3),(2.6) and (2.7) is equivalent to Dirac equations and makes the investigation of problems of motion of an electron in gravitational fields possible. Tensors C_i and C_{pq} entering these equations are, however, complex and have, apparently, no direct physical sense.

We have shown previously, that a closed system of equations in terms of tensors $\Omega, j^p, \ldots, N^{ijkl}$ can be obtained from spinor equations. When compared with complex equations, such a system offers the advantage, that the tensors entering it have a known physical meaning and that the number of unknowns is less by one (the phase φ).

In order to obtain Dirac equations in terms of tensors $\Omega, \ldots, N^{ijkl}$, we shall insert the identity

$$\psi^m = \frac{\psi^{\nu^* m}}{\sqrt{\psi^{\nu^*\nu}}} \exp(i\varphi_\nu)$$

into them, obtaining

$$\gamma^{\ k}_{nm}\, \psi^{\nu^* m} \frac{\partial \varphi^\nu}{\partial x^k} = i\left[\gamma^{\ k}_{nm}\left(\frac{\partial \psi^{\nu^* m}}{\partial x^k} - \frac{1}{2}\,\frac{\psi^{\nu^* m}}{\psi^{\nu^*\nu}}\,\frac{\partial \psi^{\nu^*\nu}}{\partial x^k}\right) + \left(\frac{ie}{\hbar c}\,\gamma^{\ k}_{nm}\,A_k + \frac{mc}{\hbar}\,\delta_{nm}\right)\right] \psi^{\nu^* m} \quad (2.12)$$

which, in terms of φ_ν and $\psi^{\nu^* n}$ is equivalent to the Dirac system. Let us denote the right-hand side of (2.12) by $P^{\nu n}$. Then

$$\gamma^{\ k}_{nm}\, \psi^{\nu^* m} \frac{\partial \varphi_\nu}{\partial x_k} = P^{\nu n} \quad (2.13)$$

from which we obtain the following independent relationships for $P^{\nu n}$:

$$\mathrm{Re}\,[\psi^{l\nu}\,\gamma_{l\ n}^{\ 4} P^{\nu n}] = 0, \qquad \mathrm{Re}\,[\psi^{l\nu}\,(\gamma^1\gamma^2\gamma^3)_{ln} P^{\nu n}] = 0$$
$$\mathrm{Re}\,[\psi^{\lambda l}\,(\gamma^3\gamma^1)_{ln}\, P^{\nu n}] = 0, \qquad \mathrm{Im}\,[\psi^{\lambda l}\,(\gamma^3\gamma^1)_{ln}\, P^{\nu n}\,] = 0 \quad (2.14)$$

It can be shown that no other equations of the type of (2.14), i.e. linear combinations of $P^{\nu n}$ exist, which are equal to zero. Relations (2.14.1) and (2.14.2) can be transformed into

$$1/_6\,\delta^{pqrn}_{k\,ij\,l}\nabla^k S^{ijl} = \frac{2mc}{\hbar}\, N^{pqrn}, \qquad \nabla_k j^k = 0 \quad (2.15)$$

and (2.15.1) can be written as

$$\nabla_k S^k = \frac{2mc}{\hbar}\, N, \qquad S^k = 1/_6\,\varepsilon^{kijl} S_{ijl}, \qquad N = 1/_{24}\,\varepsilon^{ijkl} N_{ijkl}$$

Relations (2.14.3) and (2.14.4) yield the fact that the components Φ^{314} and Φ^{234} of the tensor Φ^{pqr} are equal to zero

$$\Phi^{pqr} = j^k\nabla_k S^{pqr} + \frac{2mc}{\hbar}\, N^{npqr} j_n + 1/_6\,\delta^{npqr}_{i\,jkl}\,(S^{ijk}\nabla^l j_n - S^{ijk}\nabla_n j^l + j^l\nabla_n S^{ijk}) -$$
$$- 1/_2\,\delta^{pqr}_{ki\,j}\,(M^{ij}\nabla^k\Omega - \Omega\nabla^k M^{ij}) + N^{npqr}\nabla_k M_n^{\ k} - M_n^{\ k}\nabla_k N^{npqr} \quad (2.16)$$

and we can see that, in general, $\Phi^{pqr} = 0$ for any p, q and r.

To obtain the remaining differential equations we shall have to solve (2.13) with respect to $\partial\varphi_\nu / \partial x^p$

$$\Omega\psi^{\nu^*\nu} \frac{\partial\varphi_\nu}{\partial x_p} = \mathrm{Re}\,[\psi^{e^*\nu}(\gamma^4\gamma^p)_{ln} P^{\nu n}], \qquad \Omega\psi^{\nu^*\nu}\frac{\partial\varphi_\nu}{\partial x_4} = i\,\mathrm{Im}\,[\psi^{e^*\nu} J_{ln} P^{\nu n}]$$

$$N\psi^{\nu^*\nu} \frac{\partial\varphi_\nu}{\partial x_p} = -\,i\,\mathrm{Im}\,[\psi^{e\nu}(\gamma^5\gamma^4\gamma^p)_{ln}\, P^{\nu n}], \qquad N\psi^{\nu^*\nu}\frac{\partial\varphi_\nu}{\partial x_4} = -\,\mathrm{Re}\,[\psi^{e\nu}\gamma_{ln}^{\ 5} P^{\nu n}] \quad (2.17)$$

Transforming the right-hand sides of these equations, we obtain

$$\psi^{\nu^*\nu}\Omega\,\frac{\partial\varphi_\nu}{\partial x^p} = \frac{\psi^{\nu^*\nu}}{2}\left(\frac{\partial}{\partial x^l}\, M_p^{\ l} + \frac{2mc}{\hbar}\, j_p - \frac{2e}{\hbar c}\,\Omega A_p\right) +$$
$$+ \frac{i}{2}\,\gamma_{lm}^{\ 4}\left(\psi^{e^*\nu}\frac{\partial}{\partial x^p}\,\psi^{\nu^* m} - \psi^{\nu^* m}\frac{\partial}{\partial x^p}\,\psi^{e^*\nu}\right) \quad (2.18)$$

$$\psi^{\nu^*\nu} N^{kpqr}\frac{\partial\varphi_\nu}{\partial x^k} = -\,\frac{\psi^{\nu^*\nu}}{2}\left(\frac{1}{2}\,\delta^{pqr}_{i\,j\,k}\nabla^i M^{jk} + \frac{2e}{\hbar c}\, N^{kpqr} A_k\right) -$$
$$-\,\frac{i}{2}\,\varepsilon^{npqr}\,(\gamma^5\gamma^4)_{lm}\left(\psi^{e^*\nu}\frac{\partial}{\partial x^n}\,\psi^{\nu^* m} - \psi^{\nu^* m}\frac{\partial}{\partial x^n}\,\psi^{e^*\nu}\right)$$

Conditions of compatibility of these systems which we shall consider as algebraic equations in $\partial\varphi_\nu / \partial x^p$, give

$$N^{kpqr}\left(\nabla_l M_k{}^l + \frac{2mc}{\hbar} j_k\right) + \frac{1}{2}\Omega\delta^{pqr}_{ijk}\nabla^i M^{jk} + {}^1/_6\delta^{npqr}_{ijlk} j^k \nabla_n S^{ijl} = 0 \qquad (2.19)$$

Approximate equation analogous to (2.19) was obtained by De Broglie [9] under the assumption of the absence of external fields and of small velocity of the electron.

Last identity of (1.8) implies that (2.19) and Equation $\Phi^{pqr} = 0$, are interdependent.

To complete the tensor equations in terms of components of Ω we can use the equations of simultaneity of the system (2.17) considered as differential equations in $\partial\varphi_\nu / \partial x^p$, i.e. Equations

$$\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^k}\varphi_\nu - \frac{\partial}{\partial x^k}\frac{\partial}{\partial x^i}\varphi_\nu = 0$$

Energy — impulse tensor is however found to be more suitable for our purpose.

4. Energy — impulse tensor in the Dirac theory We know [4] that the energy — impulse tensor $T_p{}^q$ can, in the Dirac theory, be written as

$$T_p{}^q = \frac{\hbar c}{2}(\gamma^4\gamma^q)_{mn}\left(\bar\psi^m \frac{\partial\psi^n}{\partial x^p} - \psi^n\frac{\partial\bar\psi^m}{\partial x^p}\right) + \frac{e}{c}A_p j^q \qquad (2.20)$$

and the tensor thus defined, satisfies [4] Equations

$$\nabla_i T^{ki} = -e j_i H^{ki}, \qquad T^{ik} - T^{ki} = \frac{\hbar c}{2}\nabla_p S^{ikp} \qquad (2.21)$$

It can easily be shown that the following identity exists:

$$\psi^{\nu\cdot\theta}(\bar\psi^\beta d\psi^\rho - \psi^\rho d\bar\psi^\beta) \equiv \bar\psi^{\nu\beta}d\psi^{\theta\rho} - \psi^{\nu\cdot\rho}d\psi^{\beta\theta} \qquad (2.22)$$

which yields

$$\psi^{\nu\cdot\theta}T_p{}^q = \frac{\hbar c}{2}(\gamma^4\gamma^q)_{mn}\left(\bar\psi^{\nu m}\frac{\partial}{\partial x^p}\psi^{\theta n} - \psi^{\nu\cdot n}\frac{\partial}{\partial x^p}\psi^{m\cdot\theta}\right) + \frac{e}{c}A_p j^q\psi^{\nu\cdot\theta}$$

The latter, on contraction with the matrices γ^4 and $\gamma^4\gamma^n$ and with (1.10) taken into account, gives

$$T_p{}^q = \frac{\hbar c}{4\Omega}\left[\frac{1}{2}(\bar c^k\nabla_p c_k{}^q - \bar c_k{}^q\nabla_p c^k) + i\nabla_p(\Omega j^q) - j^k\nabla_p M_k{}^q - \frac{1}{6}N^{qijk}\nabla_p S_{ijk}\right] + eA_p j^q$$

$$j^n T_p{}^q = \frac{\hbar c}{4}\left\{\frac{i}{2}\left[\left(\bar c_i\nabla_p c^i + \frac{1}{2}\bar c_{ij}\nabla_p c^{ij}\right)g^{qn} - (\bar c^n\nabla_p c^q - \bar c^q\nabla_p c^n) + (\bar c^{nk}\nabla_p c_k{}^q + \right.\right.$$
$$\left.\left. + c^{qk}\nabla_p c_k{}^n)\right] + M^{qn}\nabla_p\Omega + \frac{1}{2}M_{kl}\nabla_p N^{klqn} - S^{kqn}\nabla_p j_k + i\nabla_p(j^q j^n)\right\} + eA_p j^q j^n \qquad (2.23)$$

Obviously Formula

$$j^n T_p{}^q = \frac{\hbar c}{4}\left\{\frac{i}{2}(\bar c^{nk}\nabla_p c_k{}^q + c^{qk}\nabla_p c_k{}^n) + i\nabla_p(j^q j^n) + ie^{rkqn}j_r\nabla_p j_k\right. \qquad (2.24)$$

is valid for the energy — impulse tensor in case of the neutrino.

Utilizing Equation $j^p j_p = 0$,we obtain from (2.24)

$$j_n T_p{}^n = 0 \qquad (2.25)$$

Considering the identity (2.25) as a system of equations in j_n we find, that by virtue of existence of a nonnull solution of j_n, the identity

$$\det T_p{}^q = 0 \qquad (2.26)$$

should hold.

From (2.25) we can obtain a solution $j_n = \zeta P_n$, where ζ is an arbitrary function, and P_n is a third order minor of the matrix $T_p{}^q$ obtained from it by striking out the nth row and any column. By the isotropy of the vector j_n, the following identity should be fulfilled

$$p^n P_n = 0 \tag{2.27}$$

The Lagrangian in tensor form can be obtained from (2.23) since, as we know, the Lagrangian L is given in terms of the energy – impulse tensor by

$$L = T^p{}_p + mc^2\Omega \tag{2.28}$$

Let us now obtain the expression for the components of T_p^q in terms of components of real tensors Ω, j^p, M^{ij}, S^{ijk} and N^{ijkl}. Putting

$$\psi^m = \frac{\psi^{\nu^* m}}{\sqrt{\psi^{\nu^*\nu}}} \exp(i\varphi_\nu)$$

into (2.20), we obtain $T_p{}^q$ in the following form

$$\frac{1}{c} T_p{}^q = \frac{\hbar}{2}(\gamma^4\gamma^q)_{mn}\left[\frac{1}{\psi^{\nu^*\nu}}\left(\psi^{m^*\nu}\frac{\partial\psi^{\nu^* m^*}}{\partial x^p} - \psi^{\nu^* n}\frac{\partial\psi^{m^*\nu}}{\partial x^p}\right) + 2i\psi^{m^* n}\frac{\partial\varphi_\nu}{\partial x^p}\right] + \frac{e}{c}A_p j^q \tag{2.29}$$

which, on multiplying by $\psi^{\nu^*\nu}$ and subsequent contraction with γ^4 over ν, becomes

$$\begin{aligned}1/c\Omega T_p{}^q = {}^1\!/_4\,\hbar\,[j_l\nabla_p M^{ql} - M^{ql}\nabla_p j_l - {}^1\!/_6 N^{qijk}\nabla_p S_{ijk} +\\ + {}^1\!/_6\, S_{ijk}\nabla_p N^{qijk}] + mc j_p j^q + \hbar/2\, j^q\nabla_l M_p{}^l\end{aligned} \tag{2.30}$$

Use of the last identity of (1.7) leads to the final form

$$T_p{}^q = \frac{mc^2}{\Omega} j_p j^q + \frac{\hbar c}{2\Omega}\left[j_l\nabla_p M^{ql} + j^q\nabla_l M_p{}^l - \frac{1}{6}N^{qijk}\nabla_p S_{ijk}\right] \tag{2.31}$$

and calculation of the trace of the energy – impulse tensor, yields

$$T_p{}^p = mc^2\left(\frac{j^p j_p}{\Omega} - \frac{1}{24}\frac{N^{ijkl}N_{ijkl}}{\Omega}\right) = -mc^2\Omega$$

where we have used the identity (1.7.1).

We know, that in quantum mechanics the magnitude $m\Omega$ denotes the actual mass of an electron, hence $T^p{}_p$ represents the actual energy of an electron. We notice that the form in which the tensor is given in (2.23) differs radically from that in (2.31). This is explained by the fact, that in the derivation of (2.31) fulfilment of Dirac equations was assumed, therefore the Lagrangian L formed according to Formula $L = T^p{}_p + mc^2\Omega$ becomes identically zero, while equating to zero of the Lagrangian obtained from the tensor $T_p{}^q$ in the form given in (2.23), leads to another tensor equation.

This makes it clear that the three equations (2.20.1) in which the components of $T_p{}^q$ are given in terms of components of $\Omega, j^p, \ldots, N^{ijkl}$ according to Formula (2.31) form, together with equations of (2.19), a complete system of differential equations, which can be closed by addition of nine independent algebraic equations (1.7). Another complete system can be formed from Equations (2.15), two equations of (2.19) and three equations of (2.21.1) or equations of simultaneity of the system (2.17), are second order differential equations.

The fact that three second order equations are necessary arises not from the peculiarity of our method, but from the invariance of the gradients of Dirac equations.

Indeed, a system of differential equations in tensor form should also be invariant under the gradient operation, and the fact that tensors entering these equations are invariant under gradient transformations implies, that tensor equations should contain not the potentials of external fields A_k, but the fields themselves $\nabla_p A_k - \nabla_k A_p$.

Since potentials A_k enter Dirac equations without their derivatives, hence tensor equations containing the fields should be of second order, when the terms $\partial\psi^*/\partial x^k$ are present in them. As there are three independent components of A_k, we can have three such equations.

Using the methods given above, we can easily write also nonlinear equations which would be a generalization of Dirac theory in tensor form. Such equations in most cases have the form [10]

$$\gamma^k \partial_k \psi + J\psi = 0$$

and they can be obtained in tensor notation by substituting J for $mc/\hbar$ in the Dirac equations in tensor form.

Appendix

A.1. Extension of spinor representation over the complete affine group. Let us consider a k-parameteric group G of transformations of coordinates of the n-dimensional Euclidean space R_n.

We shall choose the parameters $\alpha^1, \alpha^2, \ldots, \alpha^k$ defining the elements of the group in such a manner, that their null values define the unit element of G, and we shall consider the matrix representation of the group G in the p-dimensional space L_p.

We know that representation of such groups can be described in terms of its infinitesimal operators I_s which are defined as partial derivatives of the matrix of representation with respect to parameters of G, taken at null values of these parameters. Infinitesimal operators which appear as p-dimensional matrices, are given by

$$I_j I_m - I_m I_j = c^i_{jm} I_i \tag{A.1.1}$$

Summation is performed from 1 to k, over i. Coefficients c^i_{js} are defined by the structure of G, according to well-known formulas [11].

Let us replace the parameters $\alpha^1, \alpha^2, \ldots, \alpha^k$ with $\theta^1, \theta^2, \ldots, \theta^r$ defining some subgroup Λ of G in such a manner that $\alpha^i = \alpha^i(\theta^1, \theta^2, \ldots, \theta^r)$ and $\alpha^i(0, 0, \ldots, 0) = 0$.

Some subgroup in T will correspond to $\Lambda \subset G$. It can easily be shown that the infinitesimal operators I_s' of the representation of Λ can be defined in terms of I_s as follows

$$I_m' = I_k \left(\frac{\partial \alpha^k}{\partial \theta^m} \right)_{\theta^1 = \theta^2 = \cdots = \theta^r = 0} \tag{A.1.2}$$

Let the following representation of the matrix group be given

$$\begin{Vmatrix} 1+\alpha^1 & \alpha^2 \\ \alpha^3 & 1+\alpha^4 \end{Vmatrix}$$

where α^1, α^2, α^3 and α^4 are arbitrary parameters. Computation of the coefficients c^i_{js} results in this case in the following set of relationships for infinitesimal operators

$$I_1 I_2 - I_2 I_1 = I_2, \quad I_2 I_3 - I_3 I_2 = I_1 - I_4, \quad I_1 I_3 - I_3 I_1 = -I_3, \tag{A.1.3}$$
$$I_2 I_4 - I_4 I_2 = I_2, \quad I_1 I_4 - I_4 I_1 = 0, \quad I_3 I_4 - I_4 I_3 = -I_3$$

We shall attempt to define the representation T of the group G, which would coincide, on the orthogonal subgroup $O \subset G$, with the spinor representation of the subgroup O with its infinitesimal operator K known.

To effect the transition from G to O, we must put

$$\alpha^1 = \cos\theta - 1, \quad \alpha^2 = -\sin\theta, \quad \alpha^3 = \sin\theta, \quad \alpha^4 = \cos\theta - 1$$

Then, from (A.1.2) it follows that:

$$K = I_2 - I_3 \tag{A.1.4}$$

Since the spinor representation of weight $\frac{1}{2}$ is given by the matrices

$$\begin{Vmatrix} \exp(^1/_2 i\theta) & 0 \\ 0 & \exp(-^1/_2 i\theta) \end{Vmatrix}$$

the operator K has the form

$$K = \begin{Vmatrix} ^1/_2 i & 0 \\ 0 & -^1/_2 i \end{Vmatrix}$$

and the system (A.1.3) with the condition (A.1.4) can, after eliminating the dependent equations, be written as

$$I_1 I_3 - I_3 I_1 = -I_3, \quad I_3 K - K I_3 = \lambda J - 2I_1, \quad I_1 K - K I_1 = 2I_3 + K \qquad \text{(A.1.5)}$$

where λ is arbitrary.

It is easy to show that the system (A.1.5) is not consistent at all possible values of λ. Consequently, we find that representation of a complete group of coordinate transformations coinciding with a spinor representation on the orthogonal subgroup, does not exist. This was proved for the spinor representation of a group of rotations of a plane, but it is obviously valid for spinor representations in space of any dimension. Hence it follows that the spinor considered as a linear geometrical entity, can be introduced only into orthogonal coordinate systems. Using this method we can also show that even an increase in the number of components of the spinor does not result in the possibility of its introduction into nonorthogonal coordinate systems.

A.2. Tensors C_i, C_{pq}, j^p, M^{ij} and S^{ijk} in canonic form. Let us consider tensor aggregates $\{j^p, M^{ij}, S^{ijk}\}$ and $\{C_i, C_{pq}\}$.

We can always select such an orthogonal coordinate system, in which the components of j^p have the form

$$j^p = (0, 0, 0, i\rho) \qquad \text{(A.2.1)}$$

where we have, by virtue of (1.7.1), $\rho^2 = \Omega^2 - N^2$. From (1.7.3) it follows that in this coordinate system the components S^4 of the vector S^p becomes equal to zero.

Further, we shall perform an orthogonal transformation of the spatial coordinates x^1, x^2 and x^3 so, as to make the components of S^1 and S^2 equal to zero. Then, the components (A.2.1) will remain the same, while the components of the vector S^p will, by (1.7.2), be written as

$$S^p = (0, 0, i\rho, 0) \qquad \text{(A.2.2)}$$

From (1.7) and (1.10) it follows, that in the coordinate system just obtained, components of the tensor M^{ij} will be given by

$$M^{ij} = \begin{Vmatrix} 0 & -\Omega & 0 & 0 \\ \Omega & 0 & 0 & 0 \\ 0 & 0 & 0 & N \\ 0 & 0 & -N & 0 \end{Vmatrix} \qquad \text{(A.2.3)}$$

From the identity (1.9) we find, that in this coordinate system components of the spinor ψ^* are given as

$$\psi^1 = \psi^3 = 0, \qquad \psi^2 = \pm \sqrt{^1/_2 \rho}\, e^{i\varphi}, \qquad \psi^4 = \frac{^1/_2(\Omega + N)}{\pm\sqrt{^1/_2 \rho}}\, e^{i\varphi}$$

and remembering that $\mathrm{mod}\, \frac{\Omega + N}{\rho} = 1$, we can put

$$\frac{\Omega + N}{\rho} e^{2i\varphi} = e^{^1/_2 i\theta} \qquad \text{(A.2.4)}$$

On rotation of the plane $x^1 x^2$ by an angle θ, matrix of transformation of spinors becomes

$$S = \begin{Vmatrix} \exp(-^1/_2 i\theta) & 0 & 0 & 0 \\ 0 & \exp(^1/_2 i\theta) & 0 & 0 \\ 0 & 0 & \exp(-^1/_2 i\theta) & 0 \\ 0 & 0 & 0 & \exp(^1/_2 i\theta) \end{Vmatrix}$$

Let us now rotate the x^1x^2 plane by an angle $(-\theta)$. Then, by (1.3), which define the components of C_i and C_{pq} in terms of components of ψ, it follows, that in the obtained coordinate system, components of C_i and C_{pq} are given in the form

$$C_i = (-i\rho, \rho, 0, 0), \qquad C_{pq} = \begin{Vmatrix} 0 & 0 & N & -i\Omega \\ 0 & 0 & iN & \Omega \\ -N & -iN & 0 & 0 \\ i\Omega & -\Omega & 0 & 0 \end{Vmatrix}$$

while the components (A.2.1) to (A.2.3) remain unchanged.

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